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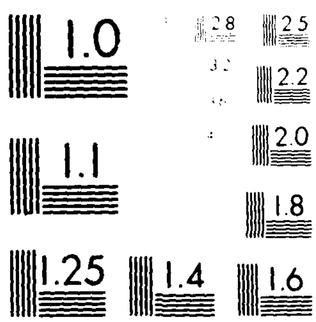

MICROCOPY RESOLUTION TEST CHART

ONE-SIDED COMPARISONS FOR TREATMENTS WITH A CONTROL[1]

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ABSTRACT: Distribution theory for likelihood ratio statistics for the comparison of several treatments with a control is discussed. These test statistics account for prior information that the treatments are at least as effective as the control. It is assumed that the sample sizes on the treatments are (approximately) equal and the sample size on the control is at least as large. Normal means are compared under the assumption of a common variance, either known or unknown. The analogous problem for proportions is also considered.

Key words and phrases: Order restricted inference, comparison with a control, chi-bar-squared distribution, E-bar-squared distribution.

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1. INTRODUCTION AND SUMMARY

We consider an experimental situation where one wishes to compare several treatments with a control or standard. For example, in a drug study, several drugs may be compared with a zero dose control. In the absence of prior information about the expected responses to these treatments one could use the results of Dunnett (1955,1964). On the other hand, we consider a situation where the investigator, because of known properties of the treatments, wishes to carry out a test making use of prior information that all of the treatment means are at least as large as the control mean. (The case in which all of the treatment means are no larger than the control mean is included by changing the signs of all the means.) Assuming that the observations are normally distributed with common variance σ^2, let H_1 denote the hypothesis that $\mu_0 \leq \mu_i$; $i = 1,2,\cdots,k$ where k is the number of treatments, μ_0 denotes the control mean and μ_i; $i = 1,2,\cdots,k$ denote the treatment means. The hypothesis, H_1, is a special type of order restriction as discussed in Barlow, Bartholomew, Bremner and Brunk (1972). The problem of testing homogeneity, H_0: $\mu_0 = \mu_i$; $i = 1,2,\cdots,k$ when the alternative is restricted by a partial order is discussed in Barlow et al. (1972) and the problem of testing a partial order as a null hypothesis is discussed in Robertson and Wegman (1978). The appropriate null hypothesis distributions for the likelihood ratio test, in each of these problems, are mixtures of standard distributions. The specific form of the distribution depends upon whether H_0 or the order restriction is the null hypothesis and upon whether or not σ^2 is assumed to be known. However, the mixing coefficients in each case are the probabilities that the maximum likelihood estimates, subject to the restriction, have a specified number of distinct values. These probabilities depend upon the order restriction and upon the sample sizes and they can be difficult, if not impossible, to compute.

The partial order, H_1, considered in this paper, is termed a tree in Barlow et al. (1972). The mixing coefficients can be obtained by numerically integrating their (3.38) and using their recursive relation, (3.23), but this approach is quite complicated for even moderate values of k. For the case where the sample sizes are equal and the total number of means does not exceed 12, the mixing coefficients are given in Table A.6 in Barlow et al. It is quite common to have significantly more observations on the control than on the treatments and we borrow an idea from Chase (1974) to find a simple approximation in this important case.

Williams (1971,1972) and Chase (1974) considered the case in which the researcher is willing to assume not only that the treatment means are greater than the control, but that the ordering among the treatment means is completely known. Thus the order restriction on the $k+1$ means is total. One application they had in mind was the comparison of increasing dosage levels of a drug with zero dose control. We follow Chase's approach and obtain approximate critical values for the tree ordering with an increased sample size on the control. The limiting values of the mixing constants are found as the sample size on the control becomes infinite. The critical values based upon these limiting constants and the critical values based upon the equal-weights mixing constants are obtained. The approximate critical values are constructed by interpolating between these two values. P-values are computed by interpolating between the P-values obtained from the equal weights and the ones obtained from the limiting values.

Siskind (1976) and Grove (1980) observed that, for total orders, the mixing constants are fairly robust to moderate variation in sample sizes. The same kind of robustness holds for the tree partial order and so the results given in this paper provide reasonable approximations even when the sample sizes on the treatments are not exactly the same.

2. ONE-SIDED TESTS

In order to be specific, consider testing the hypothesis, H_0, against the alternative $H_1 - H_0$ (i.e., H_1 but not H_0) where H_0 and H_1 are specified in the previous section. The data is obtained by choosing $k+1$ independent random samples; applying the control to one (of size n_0) and a different treatment to each of the remaining k samples (of sizes n_1, $n_2, \cdots, n_k$). Assume that the resulting sample means,

$\bar{X}_i \sim n(\mu_i, \sigma^2/n_i);\ i = 0,1,2,\cdots,k.$

We first consider the case in which σ^2 is known. The likelihood ratio test (LRT) statistic is

$$T_{01} = -2 \ln \Lambda = \Sigma_{i=0}^{k} n_i(\bar{\mu}_i - \hat{\mu})^2$$

where $\hat{\mu} = \Sigma_{i=0}^{k} n_i \bar{X}_i / \Sigma_{i=0}^{k} n_i$ and $(\bar{\mu}_0, \bar{\mu}_1, \cdots, \bar{\mu}_k)$ are the maximum likelihood estimates (MLEs) under the restriction H_1. Barlow et al. (1972) discuss the computation of the restricted MLEs for an arbitrary partial order. One algorithm that can be applied for any partial order is the minimum lower sets algorithm and this algorithm takes a very simple form for our tree ordering. Of course, if $\bar{X}_0 \le \bar{X}_i$ for $i = 1,2,\cdots,k$ then $\bar{\mu}_i = \bar{X}_i$ for $i = 0,1,2,\cdots,k$. Otherwise, arrange in increasing order the treatment sample means (the control mean is not to be included) and denote them by $\bar{X}_{(1)} \le \bar{X}_{(2)} \le \cdots \le \bar{X}_{(k)}$. Next, find the smallest positive integer j for which

$$A_j = (n_0\bar{X}_0 + \Sigma_{i=1}^{j} n_{(i)}\bar{X}_{(i)})/(n_0 + \Sigma_{i=1}^{j} n_{(i)}) < \bar{X}_{(j+1)} \tag{1}$$

where the symbol $n_{(i)}$ is used to denote the sample size associated with $\bar{X}_{(i)}$. Such an integer will exist unless $A_{k-1} \ge \bar{X}_{(k)}$ and in this case set $j = k$. Now, $\bar{\mu}_0 = A_j$ and the restricted MLE for the treatment mean is either A_j or the treatment sample mean depending on whether the treatment sample mean is included in $\Sigma_{i=1}^{j} n_{(i)}\bar{X}_{(i)}$ or not. We illustrate the algorithm with an example.

Example. Suppose $k = 3$, $n_0 = 25$, $n_1 = n_2 = n_3 = 10$, $\bar{X}_0 = 12.2$, $\bar{X}_1 = 13.1$, $\bar{X}_2 = 10.8$ and $\bar{X}_3 = 11.9$. Since the sample means do not satisfy the restrictions in H_1 we need to apply the algorithm to obtain the restricted MLEs. Clearly $\bar{X}_{(1)} = 10.8$, $\bar{X}_{(2)} = 11.9$, $\bar{X}_{(3)} = 13.1$ and $A_1 = 11.8$. Because $A_1 < \bar{X}_{(2)}$, $j = 1$ and $\bar{\mu}_0 = \bar{\mu}_2 = 11.8$, $\bar{\mu}_1 = 13.1$ and $\bar{\mu}_3 = 11.9$.

Returning to the test being considered and appealing to Theorem 3.1 of Barlow et al. (1972), we see that

$$P[T_{01}/\sigma^2 \geq c] = \Sigma_{\ell=1}^{k+1} P(\ell, k+1) P[\chi^2_{\ell-1} \geq c] \tag{2}$$

where χ^2_ν denotes a standard chi-squared variable with ν degrees freedom ($\chi^2_0 \equiv 0$) and $P(\ell, k+1)$ is the probability, under H_0, that $\bar{\mu} = (\bar{\mu}_0, \bar{\mu}_1, \cdots, \bar{\mu}_k)$ has exactly ℓ distinct values. The $P(\ell, k+1)$ depend on the values of $n_0, n_1, \cdots, n_k$. In general, for $k \leq 3$ the $P(\ell, k+1)$ can be obtained by the explicit formulas discussed on page 146 of Barlow et al. (1972) and for $k \geq 4$ they can be obtained by their recursive relation (3.23) and repeated numerical integration of their (3.38). Even for moderate k this may require several numerical integrations. If $n_0 = n_1 = n_2 = \cdots = n_k$ the values of $P(\ell, k+1)$ are given in Table A.6 of Barlow et al. (1972) for $k \leq 11$. If the sample sizes are not all equal but do not vary dramatically (say the ratio of the largest to the smallest is ≤ 2) then the values in Table A.6 provide a fairly reasonable approximation.

We are primarily interested in the case where the sample size for the control is significantly larger than the sample sizes for the treatments. In the special case in which $n_1 = n_2 = \cdots = n_k$ and n_0/n_1 is a positive integer, the tables in Ruben (1954) may be used to recursively generate the $P(\ell,k+1)$ (see the discussion on page 146 of Barlow et al. (1972)). Even in this special case their computation may be very tedious so that good approximations are of interest.

As was noted in the introduction, Chase (1974) found a good approximation for the case where n_0 is significantly larger than n_i; $i = 1,2,\cdots,k$ but when H_1 specified that $\mu_0 \le \mu_1 \le \cdots \le \mu_k$. He assumed that $n_1 = n_2 = \cdots = n_k = n$ and obtained the limiting critical values as $w = n_0/n \longrightarrow \infty$. An interpolation between the equal weights critical values and these limiting values worked very well for $1 \le w < \infty$. We employ an analogous approach for the tree ordering.

The $P(\ell,k+1)$ are computed under H_0 and so they depend only on the variances of the $\bar{X}_i$. Because the weights used in pooling the sample means to obtain $\bar{\mu}$ are the reciprocals of these variances, it is common to let $P(\ell,k+1;\, w_0,w_1,\cdots,w_k)$ denote the probability of exactly ℓ distinct values in $\bar{\mu}$ when $V(\bar{X}_i) = w_i^{-1}$ for $i = 0,1,\cdots,k$.

<u>Theorem 1</u>. If $0 < w_i < \infty$ for $i = 1,2,\cdots,k$ then

$$P(\ell,k+1;\infty,w_1,w_2,\cdots,w_k) = \lim_{w_0 \to \infty} P(\ell,k+1;w_0,w_1,w_2,\cdots,w_k) = \binom{k}{\ell-1} 2^{-k}. \quad (3)$$

<u>Proof</u>: Intuitively, as $w_0 \longrightarrow \infty$, $\bar{X}_0$ becomes degenerate at

μ_0. In the pooling process, any time one of the treatment means is amalgamated with $\bar{X}_0$ we place infinite weight on $\bar{X}_0$ so that the pooled value is equal to $\bar{X}_0$. Thus $P(\ell,k+1;\infty,w_1,\cdots,w_k)$ is equal to the probability that exactly $\ell-1$ of the treatment means are not amalgamated with $\bar{X}_0$ (i.e., exactly $\ell-1$ of the treatment means exceed the common value of $\mu_0,\mu_1,\cdots,\mu_k$). Thus, $P(\ell,k+1;\infty,w_1,\cdots,w_k) = \binom{k}{\ell-1}\left(\frac{1}{2}\right)^k$.

A more rigorous proof is obtained by induction. Obviously, for $k = 1$, $P(1,2;w_0,w_1) = P(2,2;w_0,w_1) = 1/2$ for all $0 < w_0,w_1 < \infty$. Assume the result is valid for $k = m-1$ and consider $k = m$. We use the representation for the $P(\ell,k+1)$ given in the recursive relation (3.23) in Barlow et al. (1972) (which is valid for any partial order). First some notation is needed. Let $\mathcal{L}_{\ell,k+1}$ be the collection of all partitions of $\Omega = \{0,1,\cdots,k\}$ into nonempty sets $B_1,B_2,\cdots,B_\ell$ with the B_j sets on which $\bar{\mu}$ may be constant. For $\ell \le k$, such decompositions are, with probability one, of the form $B_1 = \{0\} \cup A$ where $A \subset \{1,2,\cdots,k\}$ with $\text{card}(A) = k-\ell+1$ and $B_2,\cdots,B_\ell$ are singletons with $\bigcup_{j=2}^{\ell} B_j = \{1,2,\cdots,k\} - A$. Also set $W_{B_i} = \sum_{j\in B_i} w_j$ and for $B_i = \{j_1 < j_2 < \cdots < j_r\}$ set $w(B_i) = (w_{j_1},w_{j_2},\cdots,w_{j_r})$. For $1 < \ell \le m$, (3.23) gives

$$P(\ell,m+1:w_0,w_1,\cdots,w_m) =$$

$$\sum_{\mathcal{L}_{\ell,m+1}} P(\ell;\ell:W_{B_1},\cdots W_{B_\ell})P(1,\text{card}(B_1):w(B_1)).$$

Because $\text{card}(B_1) \le m$ if $\ell > 1$, we take the limit as $w_0 \to \infty$, apply the induction hypothesis and note that $\text{card}(\mathcal{L}_{\ell,m+1}) = \binom{m}{m-\ell+1} = \binom{m}{\ell-1}$ to obtain

$$P(\ell,m+1;\infty,w_1,\cdots,w_m) = \binom{m}{\ell-1}2^{-m} \quad \text{for} \quad \ell=2,\cdots,m.$$

Next we note that by (3.38) of Barlow et al. (1972)

$$P(m+1,m+1;w_0,w_1,\cdots,w_m) = \int_{-\infty}^{\infty}\left\{\prod_{i=1}^{m} \Phi\left(\frac{\lambda_i x}{\sqrt{1-\lambda_i^2}}\right)\right\}\phi(x)dx,$$

where Φ (ϕ) is the c.d.f. (p.d.f.) of a standard normal distribution and $\lambda_i^2 = w_i/(w_0+w_i)$ for $i=1,2,\cdots,m$. As $w_0 \to \infty$ $\lambda_i \to 0$ and so $P(m+1,m+1;\infty,w_1,\cdots,w_m) = 2^{-m}$ and $P(1,m+1;\infty,w_1,\cdots,w_m)$ is found to be 2^{-m} by subtraction. The proof is completed.

Table 1 contains the $\alpha = .1,.05$ and $.01$ critical values for the statistic T_{01}/σ^2 with $w = n_0/n = 1,\infty$ and $k=2,3,\cdots,10$. The $w = 1$ values were obtained by using (2) with the $P(\ell,k+1)$ given in the table in Barlow et al. and the $w = \infty$ values are based on the $P(\ell,k+1)$ given in Theorem 1. For $1 < w < \infty$ we recommend interpolating on $1/\sqrt{w}$. To give an indication of the accuracy of this approximation the $P(\ell,k+1;w,1,\cdots,1)$ were computed for $k=2,3,5$ and $w=2,4$. The true significance level of the test at the approximate $\alpha = .05$ cutoff value (obtained from Table 1) was then computed. The largest discrepancy for these six values was $.0028$.

Distribution theory for normal populations provides asymptotic theory under a variety of assumptions about the underlying populations (cf. Theorem 4.5 in Robertson and Wegman and Robertson (1978)). For example, one might want to compare the treatments with the control by comparing the proportions of individuals who exhibit a particular response to the stimuli. Let p_i (p_0) be the proportion of individuals in the population of interest who will exhibit the given response when treatment i (the control) is administered. We wish to test H_0: $p_0 = p_1 = \cdots = p_k$ vs. H_1-H_0 with H_1: $p_0 \le p_i$; $i = 1,2,\cdots,k$. If $\hat{p}_i$ denotes the sample proportion from a sample of size n_i and if the samples are independent then the MLEs of the p_i subject to H_1 can be obtained from the algorithm for $\bar{\mu}_i$ if the $\bar{X}_i$ are replaced by $\hat{p}_i$. Let $\bar{p}_i$; $i = 0,1,\cdots,k$, denote these restricted MLEs (cf. page 40 of Barlow et al. (1972)). Following the arguments used to prove Theorem 4.5 and Corollary 4.6 in Robertson and Wegman (1978), we see that the LRT statistic, $-2 \ln \Lambda$, is asymptotically equivalent to $T^*_{01} = [\tilde{p}(1-\tilde{p})]^{-1} \cdot \Sigma^k_{i=0} n_i(\bar{p}_i-\tilde{p})^2$ as $n_i \longrightarrow \infty$ with $n_i/n_0 \longrightarrow w_i \in (0,\infty)$ for $i = 0,1,2,\cdots,k$, where $\tilde{p} = \Sigma^k_{i=0} w_i\hat{p}_i/\Sigma^k_{i=0} w_i$. Furthermore, under the same assumptions on the sample sizes,

$$P[T^*_{01} \ge c] \longrightarrow \Sigma^{k+1}_{\ell=1} P(\ell,k+1)P[\chi^2_{\ell-1} \ge c]$$

where the $P(\ell,k+1)$ are computed with respect to the tree orde with w ghts $w_0,w_1,\cdots,w_k$. Thus T^*_{01} is a test statistic that can be used for large sample sizes and if

$n_1 = n_2 = \cdots = n_k = n$ and $n_0/n = w \geq 1$ then Table 1 can be used to obtain approximate critical values.

We now return to testing homogeneity of means versus the tree alternative, but we assume the common variance is unknown. The LRT is developed in Barlow et al. (1972) and for arbitrary sample sizes, the test statistic is a monotone function of the ratio of estimates of σ^2 under the two alternatives. Specifically,

$$S_{01} = \Sigma_{i=0}^{k} n_i(\bar{\mu}_i-\hat{\mu})^2/\Sigma_{i=0}^{k} \Sigma_{j=1}^{n_i} (X_{ij}-\hat{\mu})^2$$

where the $\bar{\mu}_i$ are the restricted MLEs (computed using the algorithm described earlier), $\hat{\mu}$ is the weighted sample mean (the sample sizes are the weights) and the X_{ij} are the actual observations. Theorem 3.2 of Barlow et al. gives the null hypothesis distribution of S_{01} as follows: for $c \geq 0$

$$P[S_{01} \geq c] = \Sigma_{\ell=1}^{k+1} P(\ell,k+1)P[B_{(\ell-1)/2,(N-\ell)/2} \geq c]$$

where $B_{a,b}$ denotes a Beta distribution with parameters a and b $(B_{0,b} \equiv 0)$, $N = \Sigma_{i=0}^{k} n_i$ and the $P(\ell,k+1)$ are defined as before. It is clear that $n_0 \longrightarrow \infty$ implies that $N \longrightarrow \infty$ which in turn implies that the distribution of S_{01} becomes degenerate at zero. However, S_{01} can be written as $\Sigma_{i=0}^{k} n_i(\bar{\mu}_i-\hat{\mu})^2/(\Sigma_{i=0}^{k} n_i(\bar{X}_i-\hat{\mu})^2 + \nu S^2)$ where νS^2 is distributed as $\sigma^2\chi_\nu^2$ and is independent of the sample means and $\nu = N-k-1$.

Following Chase's work, we consider the distribution of the statistic $S_{01}^{*} = \Sigma_{i=0}^{k} n_i(\bar{\mu}_i - \hat{\mu})^2/(\Sigma_{i=0}^{k} n_i(\bar{X}_i - \hat{\mu})^2 + Q)$ where Q is fixed, independent of the sample means and $Q \sim \sigma^2\chi_\nu^2$ (ν is now a fixed positive integer, free of N). The same arguments used to show Theorem 3.2 in Barlow et al. give

$$P[S_{01}^{*} \geq c] = \Sigma_{\ell=1}^{k+1} P(\ell, k+1) P[B_{(\ell-1)/2,(k+\nu+1-\ell)/2} \geq c].$$

With ν fixed, $n_1 = n_2 = \cdots = n_k$ and $w = n_0/n_1 \longrightarrow \infty$,

$$P[S_{01}^{*} \geq c] \longrightarrow 2^{-k} \Sigma_{\ell=1}^{k+1} \binom{k}{\ell-1} P[B_{(\ell-1)/2,(k+\nu+1-\ell)/2} \geq c]. \quad (4)$$

Table 2 contains the $\alpha = .05$ and $.01$ critical values for this limiting distribution of S_{01}^{*} with $w = n_0/n = 1, \infty$, $k = 2, 3, \cdots, 10$ and $\nu = 2,4,6,8,10,12,14,16,18,20,30,50,100$. The values for $w = 1$ were obtained by using the $P(\ell, k+1)$ in Appendix A.6 of Barlow et al. and those for $w = \infty$ were obtained by using (4). As in Chase's work, we recommend interpolating on $1/\sqrt{w}$ for $1 < w < \infty$ and on $N^{-1} = (\nu+k+1)^{-1}$ for intermediate values of ν.

In many applications of the above procedures the researcher is actually interested in finding evidence in favor of H_1 rather than in finding evidence against H_0. Testing H_0 against $H_1 - H_0$ allows one to control the probability of falsely confirming H_1. If one is more interested in controlling the probability of falsely rejecting H_1 he also might consider the test procedure developed in Robertson and Wegman

(1978). They considered the likelihood ratio test where H_1 is the null hypothesis. If σ^2 is known, the test statistic is $T_{12} = -2 \ln \Lambda = \sum_{i=0}^{k} n_i(\bar{\mu}_i - \bar{X}_i)^2$. Within H_1, H_0 is least favorable and under H_0,

$$P[T_{12} \geq c] = \sum_{\ell=1}^{k+1} P(\ell, k+1) P[\chi^2_{k+1-\ell} \geq c].$$

Clearly, if $n_1, n_2, \cdots, n_k$ are fixed and $n_0 \longrightarrow \infty$ then

$$P[T_{12} \geq c] \longrightarrow \sum_{\ell=1}^{k+1} \binom{k}{\ell-1} 2^{-k} P[\chi^2_{k+1-\ell} \geq c]. \qquad (5)$$

The equal weights critical values (taken from Robertson and Wegman (1978)) and the critical values for $n_0 = \infty$ (computed from (5)) are given in Table 3. Again we recommend interpolation on $(n_0/n_1)^{-1/2}$ when $n_1 = n_2 = \cdots = n_k$ and $n_0 \geq n$.

If the common variance, σ^2, is unknown then the likelihood ratio statistic, for testing H_1 against $\sim H_1$, is given by

$$S_{12} = \frac{\sum_{i=0}^{k} n_i(\bar{X}_i - \bar{\mu}_i)^2}{\sum_{i=0}^{k} n_i(\bar{X}_i - \bar{\mu}_i)^2 + \nu S^2}$$

and the appropriate null hypothesis distribution (computed under H_0) is

$$P[S_{12} \geq c] = \sum_{\ell=1}^{k+1} P(\ell, k+1) P[B_{(k+1-\ell)/2, (N-k-1)/2} \geq c].$$

We have the same difficulty we had in testing H_0 vs. $H_1 - H_0$, in that the distribution becomes degenerate as n_0 (and thus N) gets large. We adopt an analogous remedy by fixing νS^2 and then letting $n_0/n_1 \longrightarrow \infty$ with $n_1 = n_2 = \cdots = n_k$. Under these conditions we find that the appropriate probability is $2^{-k} \sum_{\ell=1}^{k+1} \binom{k}{\ell-1} P[B_{(k+1-\ell)/2,\nu/2} \geq c]$. Again for intermediate values of $w = n_0/n_1$ and ν we recommend interpolation in Table 4 on $1/\sqrt{w}$ and on N^{-1}.

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TABLE 1. Critical values of T_{01}/σ^2 for $w = 1, \infty$.

	α					
	.1		.05		.01	
k	w = 1	w = ∞	w = 1	w = ∞	w = 1	w = ∞
2	3.275	2.953	4.577	4.231	7.672	7.283
3	4.696	4.010	6.171	5.433	9.561	8.740
4	6.036	4.955	7.654	6.500	11.295	10.020
5	7.333	5.836	9.075	7.481	12.939	11.180
6	8.600	6.672	10.456	8.411	14.523	12.275
7	9.848	7.476	11.821	9.295	16.061	13.325
8	11.081	8.257	13.136	10.153	17.564	14.300
9	12.301	9.018	14.446	10.984	19.039	15.275
10	13.510	9.764	15.741	11.800	20.490	16.200

TABLE 2. Critical values for S^*_{01} for $w = n_0/n_1 = 1,\infty$.

ν	α	w \ k	2	3	4	5	6	7	8	9	10
2	0.05	1	.8779	.9033	.9172	.9265	.9333	.9385	.9426	.9460	.9490
		∞	.8511	.8550	.8501	.8425	.8340	.8257	.8171	.8091	.8018
	0.01	1	.9727	.9775	.9800	.9817	.9829	.9839	.9849	.9854	.9861
		∞	.9653	.9619	.9561	.9482	.9404	.9316	.9238	.9150	.9072
4	0.05	1	.6650	.7300	.7700	.7983	.8193	.8359	.8491	.8601	.8694
		∞	.6328	.6724	.6924	.7031	.7890	.7119	.7134	.7136	.7129
	0.01	1	.8428	.8730	.8916	.9048	.9146	.9219	.9282	.9331	.9375
		∞	.8242	.8398	.8457	.8467	.8457	.8428	.8398	.8359	.8320
6	0.05	1	.5229	.5996	.6514	.6899	.7197	.7437	.7632	.7800	.7942
		∞	.4932	.5459	.5771	.5977	.6123	.6226	.6301	.6357	.6401
	0.01	1	.7129	.7617	.7930	.8714	.8350	.8496	.8613	.8711	.8794
		∞	.6914	.7227	.7402	.7510	.7568	.7607	.7627	.7637	.7637
8	0.05	1	.4287	.5059	.5610	.6040	.6382	.6663	.6899	.7102	.7278
		∞	.4023	.4570	.4927	.5181	.5371	.5518	.5630	.5723	.5796
	0.01	1	.6094	.6670	.7070	.7373	.7607	.7803	.7969	.8105	.8223
		∞	.5898	.6289	.6533	.6699	.6816	.6895	.6953	.7002	.7041
10	0.05	1	.3623	.4365	.4917	.5356	.5718	.6021	.6279	.6504	.6699
		∞	.3389	.3926	.4292	.4565	.4778	.4946	.5083	.5198	.5293
	0.01	1	.5303	.5898	.6348	.6680	.6953	.7187	.7383	.7549	.7686
		∞	.5117	.5547	.5820	.6025	.6172	.6289	.6387	.6455	.6514
12	0.05	1	.3132	.3833	.4370	.4805	.5171	.5483	.5752	.5991	.6199
		∞	.2925	.3437	.3799	.4077	.4299	.4480	.4631	.4758	.4868
	0.01	1	.4687	.5293	.5742	.6094	.6387	.6641	.6855	.7041	.7207
		∞	.4502	.4941	.5234	.5469	.5635	.5771	.5879	.5977	.6055
14	0.05	1	.2759	.3413	.3931	.4355	.4717	.5029	.5303	.5547	.5764
		∞	.2573	.3054	.3408	.3682	.3906	.4092	.4250	.4385	.4502
	0.01	1	.4180	.4775	.5234	.5596	.5898	.6162	.6387	.6592	.6768
		∞	.4023	.4453	.4766	.5000	.5176	.5332	.5449	.5566	.5645
16	0.05	1	.2463	.3076	.3569	.3979	.4336	.4644	.4917	.5161	.5383
		∞	.2295	.2749	.3086	.3354	.3579	.3765	.3926	.4065	.4189
	0.01	1	.3779	.4355	.4805	.5166	.5479	.5742	.5977	.6191	.6377
		∞	.3633	.4043	.4355	.4590	.4785	.4941	.5078	.5195	.5293
18	0.05	1	.2227	.2800	.3267	.3662	.4009	.4312	.4583	.4827	.5046
		∞	.2073	.2498	.2822	.3081	.3301	.3486	.3647	.3789	.3914
	0.01	1	.3447	.4004	.4434	.4795	.5107	.5381	.5615	.5830	.6016
		∞	.3301	.3711	.4014	.4258	.4453	.4609	.4756	.4873	.4980

TABLE 2 (cont.)

ν	α	w \ k	2	3	4	5	6	7	8	9	10
20	0.05	1	.2029	.2568	.3013	.3394	.3726	.4023	.4290	.4529	.4749
		∞	.1887	.2288	.2598	.2849	.3062	.3245	.3406	.3547	.3672
	0.01	1	.3164	.3691	.4121	.4473	.4785	.5049	.5293	.5508	.5703
		∞	.3027	.3428	.3721	.3955	.4160	.4316	.4463	.4590	.4697
30	0.05	1	.1406	.1816	.2166	.2476	.2754	.3008	.3242	.3459	.3661
		∞	.1306	.1611	.1858	.2065	.2249	.2410	.2554	.2686	.2805
	0.01	1	.2246	.2676	.3027	.3340	.3613	.3867	.4092	.4297	.4487
		∞	.2148	.2471	.2725	.2930	.3115	.3271	.3418	.3545	.3652
50	0.05	1	.0872	.1144	.1384	.1603	.1807	.1996	.2175	.2344	.2504
		∞	.0808	.1012	.1183	.1332	.1466	.1588	.1702	.1805	.1904
	0.01	1	.1416	.1719	.1973	.2207	.2422	.2617	.2803	.2969	.3135
		∞	.1353	.1582	.1768	.1924	.2070	.2197	.2314	.2422	.2524
100	0.05	1	.0447	.0594	.0727	.0851	.0969	.1083	.1190	.1295	.1396
		∞	.0413	.0524	.0620	.0704	.0784	.0857	.0926	.0991	.1055
	0.01	1	.0737	.0906	.1055	.1191	.1323	.1145	.1563	.1670	.1777
		∞	.0703	.0830	.0938	.1035	.1123	.1206	.1282	.1353	.1421
∞	0.05	1	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
		∞	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	0.01	1	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
		∞	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

TABLE 3. Critical values of T_{12}/σ^2 for $w = 1, \infty$.

	α					
	.1		.05		.01	
k	w = 1	w = ∞	w = 1	w = ∞	w = 1	w = ∞
2	2.5796	2.953	3.8232	4.231	6.8203	7.283
3	3.1992	4.010	4.5469	5.433	7.7344	8.740
4	3.6658	4.955	5.0830	6.500	8.4082	10.020
5	4.0474	5.836	5.5283	7.481	8.9648	11.180
6	4.3630	6.672	5.8909	8.411	9.3926	12.275
7	4.6406	7.476	6.2109	9.295	9.7969	13.325
8	4.8845	8.257	6.4863	10.153	10.1250	14.300
9	5.1013	9.018	6.7383	10.984	10.4297	15.275
10	5.3013	9.764	6.9609	11.800	10.7314	16.200

TABLE 4.

ν	α	w \ k	2	3	4	5	6	7	8	9	10
2	0.05	1	.8823	.9087	.9229	.9319	.9380	.9429	.9463	.9492	.9517
		∞	.9013	.9333	.9497	.9596	.9662	.9710	.9746	.9774	.9796
	0.01	1	.9761	.9817	.9846	.9863	.9878	.9885	.9893	.9900	.9905
		∞	.9801	.9867	.9900	.9920	.9933	.9943	.9950	.9955	.9960
4	0.05	1	.6387	.6885	.7192	.7402	.7559	.7683	.7783	.7866	.7935
		∞	.6714	.7406	.7839	.8141	.8365	.8540	.8680	.8794	.8890
	0.01	1	.8340	.8594	.8750	.8848	.8926	.8984	.9023	.9062	.9102
		∞	.8506	.8843	.9047	.9187	.9289	.9368	.9430	.9481	.9524
6	0.05	1	.4858	.5371	.5708	.5947	.6133	.6279	.6401	.6504	.6592
		∞	.5182	.5934	.6450	.6836	.7139	.7385	.7590	.7763	.7912
	0.01	1	.6934	.7285	.7500	.7656	.7773	.7871	.7939	.8008	.8066
		∞	.7138	.7631	.7958	.8197	.8381	.8529	.8650	.8752	.8839
8	0.05	1	.3896	.4375	.4697	.4932	.5117	.5264	.5391	.5498	.5591
		∞	.4190	.4911	.5432	.5839	.6169	.6445	.6680	.6884	.7062
	0.01	1	.5840	.6211	.6465	.6641	.6777	.6895	.6992	.7070	.7129
		∞	.6062	.6611	.6995	.7288	.7521	.7714	.7876	.8014	.8135
10	0.05	1	.3247	.3682	.3979	.4199	.4375	.4521	.4644	.4746	.4839
		∞	.3509	.4177	.4677	.5078	.5412	.5696	.5943	.6160	.6352
	0.01	1	.5020	.5410	.5654	.5840	.5977	.6094	.6201	.6289	.6357
		∞	.5240	.5797	.6202	.6519	.6778	.6995	.7182	.7344	.7487
12	0.05	1	.2778	.3174	.3447	.3655	.3818	.3955	.4072	.4175	.4263
		∞	.3016	.3629	.4101	.4486	.4812	.5094	.5342	.5563	.5761
	0.01	1	.4395	.4766	.5010	.5195	.5332	.5459	.5557	.5645	.5723
		∞	.4603	.5147	.5553	.5878	.6148	.6378	.6578	.6754	.6911
14	0.05	1	.2427	.2788	.3042	.3232	.3389	.3516	.3623	.3721	.3804
		∞	.2642	.3207	.3648	.4014	.4328	.4603	.4847	.5066	.5265
	0.01	1	.3906	.4258	.4492	.4668	.4805	.4922	.5029	.5117	.5195
		∞	.4099	.4622	.5020	.5343	.5615	.5850	.6057	.6242	.6404
16	0.05	1	.2156	.2485	.2720	.2896	.3042	.3162	.3264	.3354	.3433
		∞	.2351	.2871	.3284	.3630	.3930	.4195	.4433	.4648	.4844
	0.01	1	.3516	.3838	.4062	.4238	.4375	.4492	.4590	.4668	.4746
		∞	.3692	.4191	.4576	.4893	.5162	.5397	.5605	.5792	.5960
18	0.05	1	.1938	.2241	.2456	.2622	.2759	.2871	.2969	.3052	.3127
		∞	.2117	.2599	.2985	.3313	.3599	.3853	.4083	.4292	.4484
	0.01	1	.3184	.3496	.3711	.3877	.4014	.4121	.4219	.4297	.4365
		∞	.3358	.3831	.4201	.4509	.4774	.5006	.5213	.5400	.5569

TABLE 4 (cont.)

ν	α	w \ k	2	3	4	5	6	7	8	9	10
20	0.05	1	.1760	.2041	.2241	.2397	.2524	.2629	.2720	.2800	.2871
		∞	.1925	.2373	.2736	.3045	.3318	.3562	.3783	.3985	.4172
	0.01	1	.2920	.3213	.3418	.3574	.3701	.3809	.3896	.3975	.4043
		∞	.3078	.3527	.3882	.4180	.4437	.4665	.4869	.5054	.5223
30	0.05	1	.1206	.1409	.1558	.1674	.1769	.1851	.1919	.1980	.2036
		∞	.1324	.1654	.1928	.2168	.2383	.2580	.2762	.2932	.3091
	0.01	1	.2051	.2275	.2432	.2559	.2666	.2754	.2822	.2891	.2949
		∞	.2169	.2521	.2807	.3054	.3273	.3471	.3651	.3818	.3973
50	0.05	1	.0740	.0870	.0967	.1042	.1106	.1160	.1206	.1248	.1285
		∞	.0815	.1029	.1211	.1374	.1523	.1661	.1791	.1914	.2031
	0.01	1	.1279	.1436	.1543	.1631	.1704	.1768	.1816	.1865	.1904
		∞	.1361	.1601	.1802	.1979	.2139	.2287	.2424	.2553	.2675
100	0.05	1	.0376	.0444	.0496	.0536	.0571	.0599	.0625	.0648	.0699
		∞	.0415	.0529	.0627	.0716	.0799	.0878	.0952	.1024	.1092
	0.01	1	.0662	.0745	.0806	.0854	.0894	.0928	.0959	.0986	.1011
		∞	.0704	.0836	.0950	.1051	.1144	.1232	.1314	.1393	.1468
∞	0.05	1	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
		∞	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	0.01	1	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
		∞	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

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Distribution theory for likelihood ratio statistics for the comparison of several treatments with a control is discussed. These test statistics account for prior information that the treatments are at least as effective as the control. It is assumed that the sample sizes on the treatments are (approximately) equal and the sample size on the control is at least as large. Normal means are compared under the assumption of a common variance, either known or unknown. The analogous problem for proportions is also considered.

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